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DYNAMICS OF FLUID MOTION

ABOUT AN AIRFOIL

by

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Submitted in Partial Fulfillment of the Requirement
for the Degree of
MASTER OF SCIENCE

from the
Massachusetts Institute of Technology
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Signature of Author.....
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May 1928 .

Professor A. A. Merrill
Secretary of the Faculty
Massachusetts Institute of Technology

Dear Sir:

In accordance with the regulation for
graduation with the Degree of Master of Science, I
herewith submit my thesis entitled "Dynamics of Fluid
Motion About an Airfoil."

Respectfully,

✓

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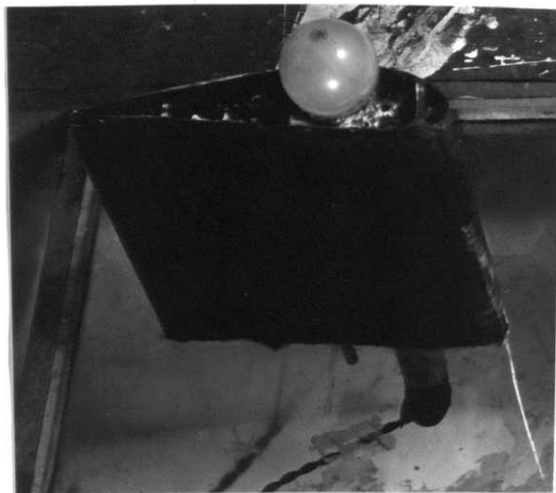
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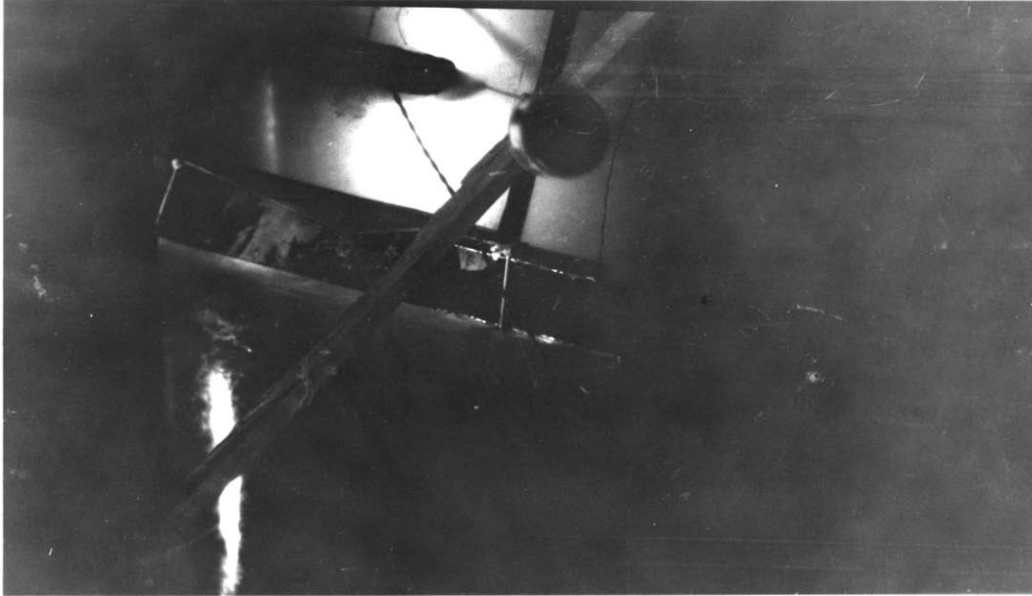
I N T R O D U C T I O N

The following discussion of the dynamics of fluid flow past an airplane wing is not intended to offer anything new to the science, except perhaps a suitable explanation of the origin of circulation about the wing, but is the line of thought followed by the author in a general qualitative study of the subject made exclusively for his own enlightenment. However, it is hoped that this discussion may be of benefit to others who may have similar difficulties in visualizing the physical conditions involved which are usually described by means of rather abstract and somewhat advanced mathematics.

In connection with this study, observations made with an actual wing section in a stream of water were found very helpful and instructive in obtaining general information about the flow and in support of general assumptions and conclusions. The attached photographs show the somewhat crude glass bottomed boat with wing section attached that was used for much of the study. The stream lines were traced easily by a solution of potassium permanganate introduced by a jet into any part of the stream desired and illuminated from underneath by an electric light. Another method found convenient for exploring the field was to hold a short thread in the stream whenever desired by means of a fine wire and its direction indicated the direction of flow. Since this



Wing Section Used for Under Water Observations.



Showing Method of Tracing Streamlines
About a Wing in a Stream of Water.

was only qualitative, and was made under very unfavorable conditions in a muddy and turbulent canal in the hydraulic laboratories of the Massachusetts Institute of Technology with very limited facilities rigged up entirely at the personal expense of the author, an attempt to obtain suitable photographs was not considered worth while and only one, snapped with a portable camera, is appended to show the general situation but not the results.

By this method, with a suitable canal or tunnel, very valuable and instructive data could easily be obtained and the solution of most all problems involved would thereby be greatly simplified. For the benefit of anyone who might wish to use a similar apparatus in water, it might well be added that with a fifty watt light, ten inches below the surface good photographs can be made in clear water any place between this depth and the surface and by using a timed intermittent jet of colored fluid, the speed of flow can thereby be accurately determined.

Observations in connection with drag were made with a jet of water projected against various kinds of surfaces and a photograph is appended of a wing with a traveling surface designed to eliminate drag on the upper surface by preventing the loss of momentum of the fluid in contact with the surface in passing from the leading to the trailing edge. This loss of momentum is so rapid that the stream from a small hose striking the surface tangentially at a speed estimated to be above fifty feet per second is

is completely stopped in passing across a ten inched curved surface. (See attached photograph). By moving the surface at the same speed, the fluid can be made to leave without loss of momentum. The other section on the same photograph was designed so that fluid could be projected from inside the wing at intervals in sheets tangentially to the surface to accomplish the same results as the traveling surface. This device was not tried out because of the unfavorable conditions under which the work had to be done.

Figures 1 and 2 are sketches of an apparatus made of small rubber tube curved over a wooden block to demonstrate the effect of centrifugal force and the theory discussed on pages 4 and 5. A photograph of a part of this apparatus is also appended to show the lift due to the change in direction of the stream.

As it is not desired to discuss corrections for wing tips etc. the following discussion will be confined to two dimension and only apply to wings of infinite span, and the viscosity of the fluid is entirely neglected up to the discussion of the drag forces.

I

AN ILLUSTRATION OF LIFT DUE TO CENTRIFUGAL FORCE.

An Illustration of Lift Due to Centrifugal Force.

A very clear illustration of the cause of lift on an airplane wing can be obtained by an analysis of an artificial flow about an unsymmetrical body with a section made up of simple curves but in general similar to an air-foil section.

Consider the body shown in figure 1 with a thin horizontal projection ahead far enough to divide the stream while it is still horizontal and imagine a flow such that the stream tubes keep a constant cross section and consequently a constant speed. The profile of the body consists of a quarter circle convex downward, a half circle convex upward and another quarter convex downward. The force against the body due to an element of mass is $m V^2/r$ g, V being the speed, and r the radius of the curve. The Total force, F , due to the mass between two stream lines would be the integral of $(D V^2/r) g ds$ around the turn where ds is an element of arc and equal $r d\theta$ and D the mass in a unit of length of the tube considered and assumed to be constant.

Then $F = DV^2/g \int_0^\pi \frac{r}{r} d\theta = DV^2\theta/g$ (1) which means that the force is independent of the radius of turn but equal to the momentum that goes through the tube per unit of time, times the angle through which its direction is changed, the angle of "downwash." The same is true for all the other stream tubes, so that the total force is equal

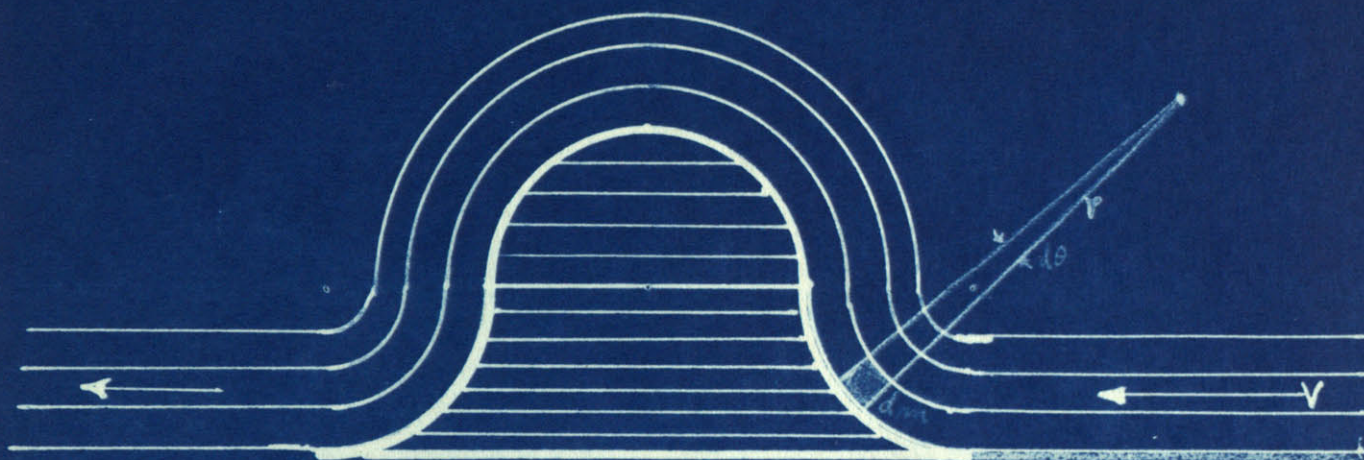


Fig. 1

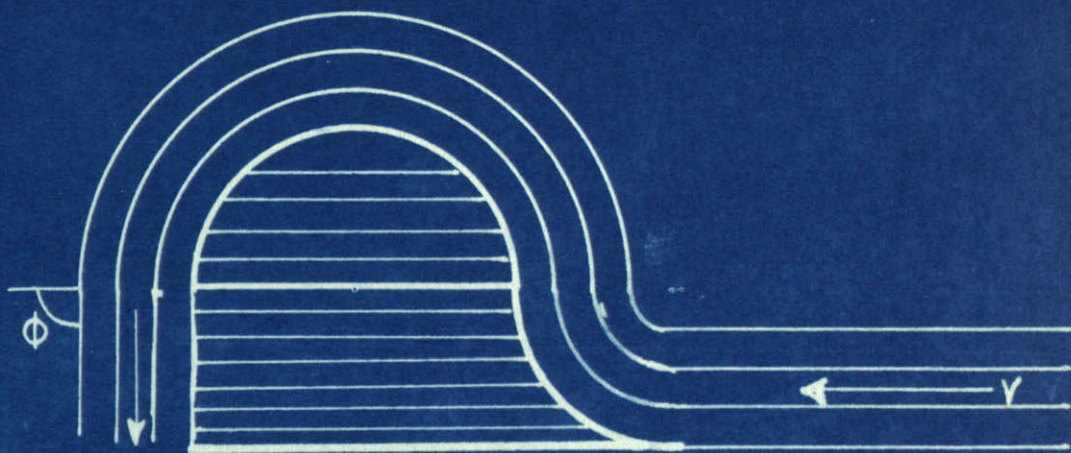


Fig. 2



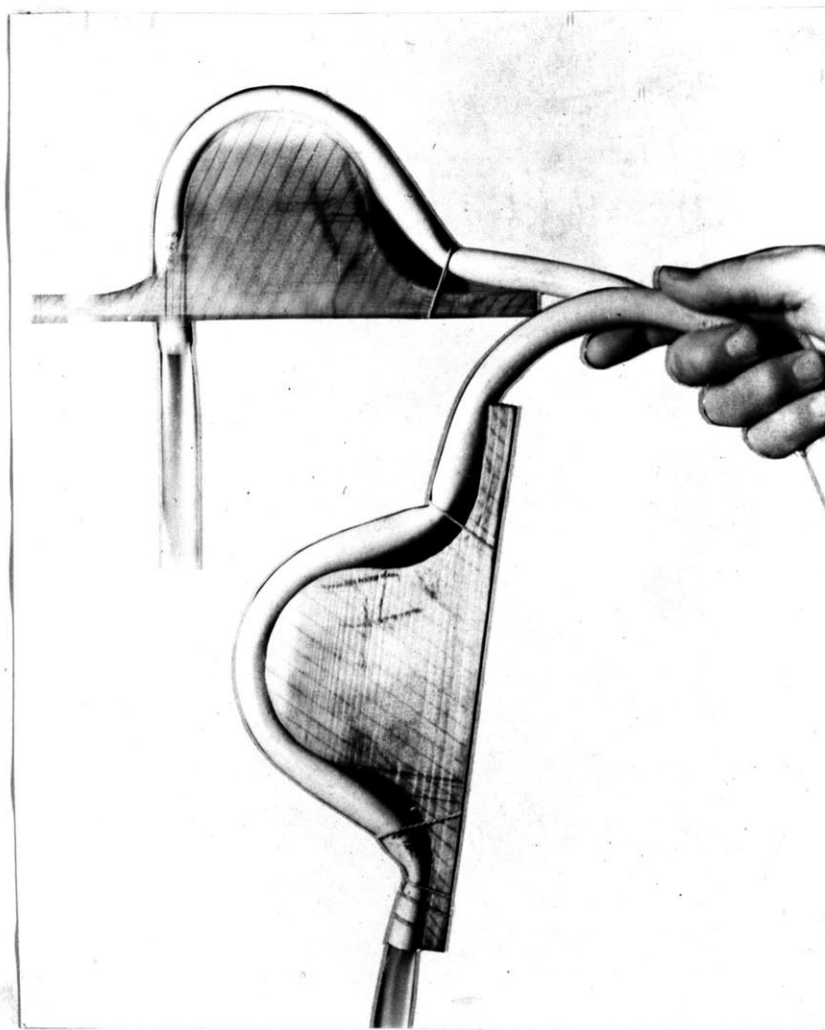
Fig. 3

rate of
to the total, angular change in momentum. It is easily seen at least for the fluid near the surface that the resultant force due to the first 90° turn is at a 45° angle to the direction of the undisturbed stream and the resultant force due to the next 90° turn is equal and opposite to the first and a similar argument shows that if the final direction is the same as the original, there is no resultant force, but if the last turn is left off as in figure 2 (see also photograph appended) there is a resultant force with components of both lift and drag and if the final angle were as in figure 3, the resultant lift would be less, but the drag would be less in proportion than the lift.

Until the cohesive strength of the fluid is reached, this lift is applied directly to the body and then is equally effective in balancing the static pressure of the fluid. If the cohesive strength plus the static pressure is insufficient, the stream leaves the surface or burbles leaving a partial vacuum temporarily until it is filled by backflow and turbulence which follows the wing.

Still assuming that the stream lines have a constant cross section and that the speed is constant, it can be seen that the horizontal stream underneath in combining with the stream along the upper surface reduces the angle of downwash for each successive stream tube or strip of fluid until at some distance above the body, it becomes negligible, See fig. 4.

For the first strip above the wing of depth Δr



Double Print to Show How Lift Results
From Changing Direction of Stream.

consider an equal strip below differing in direction of flow at the meeting point by the angle θ . The combined flow continues at an angle $= \theta/2$ below the horizontal. In making this change in direction, the top strip would of course thicken up in the angle to make a rounded curve, but if this were prevented the angle on top of the strip would be shifted back from the angle on the lower side as though the strip were rotated about the apex of the angle on the lower side through an angle $= \theta/2$ as can be seen from the figure and the angle at which the next strip would leave the curve would be reduced by an amount equal to $\frac{\theta_0}{2} \frac{\Delta r}{r_0}$ and the new angle θ would be equal to $\theta_0 (1 - \frac{\Delta r}{2r_0})$. For the next strip above combined with its corresponding strip below, the deviation would be $\theta_0 (\frac{1}{2} - \frac{\Delta r}{4r_0})$ and the two would combine with the preceding two at a common angle of $\frac{1}{2} [\frac{\theta_0}{2} + \theta_0 (\frac{1}{2} - \frac{\Delta r}{4r_0})] = \theta_0 (\frac{1}{2} - \frac{\Delta r}{8r_0})$ below the horizontal. This would rotate the top strip back again through an angle $= \theta_0 (1 - \frac{\Delta r}{2r_0}) - \theta_0 (\frac{1}{2} - \frac{\Delta r}{8r_0}) = \theta_0 (\frac{1}{2} - \frac{3\Delta r}{8r_0})$ and reduce the downwash angle of the third strip by an angle $= \theta_0 (\frac{1}{2} - \frac{3\Delta r}{8r_0}) \frac{\Delta r}{r_1} = \theta_0 \frac{\Delta r}{2r_1}$ if Δr^2 is neglected and the resulting new value of θ would be $\theta_0 [1 - \frac{\Delta r}{2} (\frac{1}{r_0} + \frac{1}{r_1})]$. This process of analysis is very tedious but has been repeated for the next two strips to make sure of the law of decrease but these will be omitted. Disregarding the term Δr^2 each time, the angle at which each successive strip leaves the curve and joins the main stream is $\theta_0 [1 - \frac{\Delta r}{2} (\frac{1}{r_0} + \frac{1}{r_1} + \frac{1}{r_2} + \dots)]$
 $= \theta_0 (1 - \sum \frac{\Delta r}{2r})$ and by taking strips of different thickness, $\theta = \theta_0 (1 - \int_r \frac{dr}{2r})$

$$= \theta_0 \left(1 - \frac{1}{2} \log \frac{r}{r_0}\right) \quad (2) \quad \text{when } r = e^2 r_0,$$

the angle of downwash is reduced to zero and at greater distances is negative, but the resultant force from this distance out is negligible and will not be considered in this approximation.

Now, as shown in formula (1) page 4 the lift is equal to $D V^2 \theta / g$ where D is the mass of a unit length of a strip or tube, but it will be more convenient now to consider D as density and to take a section of unit depth along the span and then the mass of a unit length along the tube would be $D dr$ and the net lift integrated out to where $\theta=0$ is

$$\begin{aligned} F &= D V^2 / g \int_{r_0}^{e^2 r_0} \theta dr = D V^2 / g \int_{r_0}^{e^2 r_0} \theta_0 \left[1 - \frac{1}{2} \log \frac{r}{r_0}\right] dr \\ &= D V^2 \theta_0 / g \left[r - \frac{r_0}{2} \left(\frac{r}{r_0} \log \frac{r}{r_0} - \frac{r}{r_0} \right) \right]_{r_0}^{e^2 r_0} = D V^2 \theta_0 / g \left[\frac{3}{2} r - \frac{r}{2} \log \frac{r}{r_0} \right]_{r_0}^{e^2 r_0} \\ &= D V^2 \theta_0 / g \left[\frac{3}{2} e^2 r_0 - \frac{3}{2} r_0 - e^2 r_0 \right] = D V^2 \theta_0 r_0 / g \left(\frac{e^2 - 3}{2} \right) \\ &= 2.192 D V^2 \theta_0 r_0 / g \quad (3) \end{aligned}$$

and since $r_0 \theta_0$ is the length of the arc from the highest point to the trailing edge this can easily be compared with experimental values of lift on an ordinary wing section which are usually represented as a coefficient for one foot of span and one foot of chord at one mile per hour speed. One foot of chord at zero angle of attack on a wing similar to the Clark Y or the Fokker would correspond to an arc of about .7 foot as used in the above formula. Computing the coefficient then for one mile per hour = 1.467 feet

per second in air density .07608 lbs. per cubic foot,

$$F = \frac{2.1828 \times .07608 \times .7}{32.2 \times 1.467 \times 1.467} = .001670$$

which is of the order of ~~the~~ the coefficient experimentally determined for some common wings and is at least an approximation to what might be expected since the downwash below the wing has not been considered and since in actual flow about a wing, the speed is much higher over the upper surface giving an increased lift and bringing the resultant lift vector nearer the vertical thereby reducing the drag. The lift vector as computed would be at an angle somewhere between $\theta_0/2$ and 0 behind the vertical.

In an actual wing section, the function performed by the projection forward of the wing described above and illustrated in figure 1 would be performed by the fluid beneath which would suffer an equal but opposite deviation from its direction of flow except very near the wing which would be net lift added to that considered already and it would probably be still further deflected as it passed along the wing giving still more lift and at the same time interfering less with the downwash from above. The resultant downwash angle on the surface would be equal to the angle of attack and would decrease to zero at some distance below the wing. In the discussion above, if the flow below had been considered to have been deviated by

an angle α the first value of $\Delta \theta$ would have been $\frac{\theta_0 - \alpha}{2} \frac{\Delta r}{r_0}$,

the second value $\frac{\theta_0 - \alpha}{2} \frac{\Delta r}{r_1}$ etc. and θ would have been

equal to $\theta_0 - \frac{\theta_0 - \alpha}{2} \log r/r_0$ and would become zero

when $\frac{(\theta_0 - \alpha)}{2} \log r/r_0 = \theta_0$ and $r = r_0 e^{\frac{2\theta_0}{\theta_0 - \alpha}}$

$$\text{then } F = DV^2/g \int_{r_0}^{r_0 e^{\frac{2\theta_0}{\theta_0 - \alpha}}} \left[\theta_0 dr - \frac{r_0}{2} (\theta_0 - \alpha) \log \frac{r}{r_0} d\left(\frac{r}{r_0}\right) \right]$$

$$= DV^2/g \left[\theta_0 r_0 (e^{\frac{2\theta_0}{\theta_0 - \alpha}} - 1) - \frac{(\theta_0 - \alpha)}{2} \left[r_0 e^{\frac{2\theta_0}{\theta_0 - \alpha}} \frac{2\theta_0}{\theta_0 - \alpha} - r_0 (e^{\frac{2\theta_0}{\theta_0 - \alpha}} - 1) \right] \right]$$

For a value of $\alpha = \frac{1}{2} \theta_0$,

$$F = DV^2/g (e^4 - 5) \frac{r_0 \theta_0}{4} = 12.5 DV^2 r_0 \theta_0 / g$$

which is $\frac{12.5}{2.1821} = 5.7$ times as much as previously figured for flow above the wing and to this must be added the lift due to downwashing the fluid below and the increase lift due to high speeds over the leading edge.

While the foregoing discussion does not show the actual lift, it illustrates clearly why there should be lift and indicates about the amount that could be expected.

Figure 6 shows about what the flow pattern would be. Forward of the wing would be a region of high pressure which would curve the lower stream lines downward and the upper ones upward by equal amounts except very slightly before the most forward point of the wing is reached. The pressure is the result of the integral of $dm V^2/r$ and is greatest where the curvature is greatest and the highest rate of thickening of the stream tubes is where the pressure gradient is greatest. Since the geometry gets complicated and since the radius of curvature increases for successive strips toward the center of curvature a quantitative determination of the lift forces by direct integration

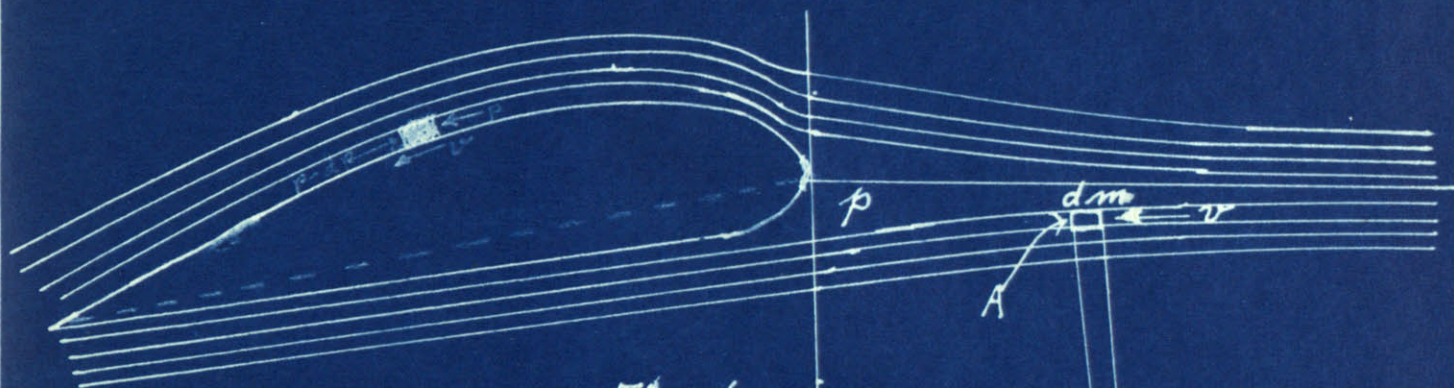


Fig. 6

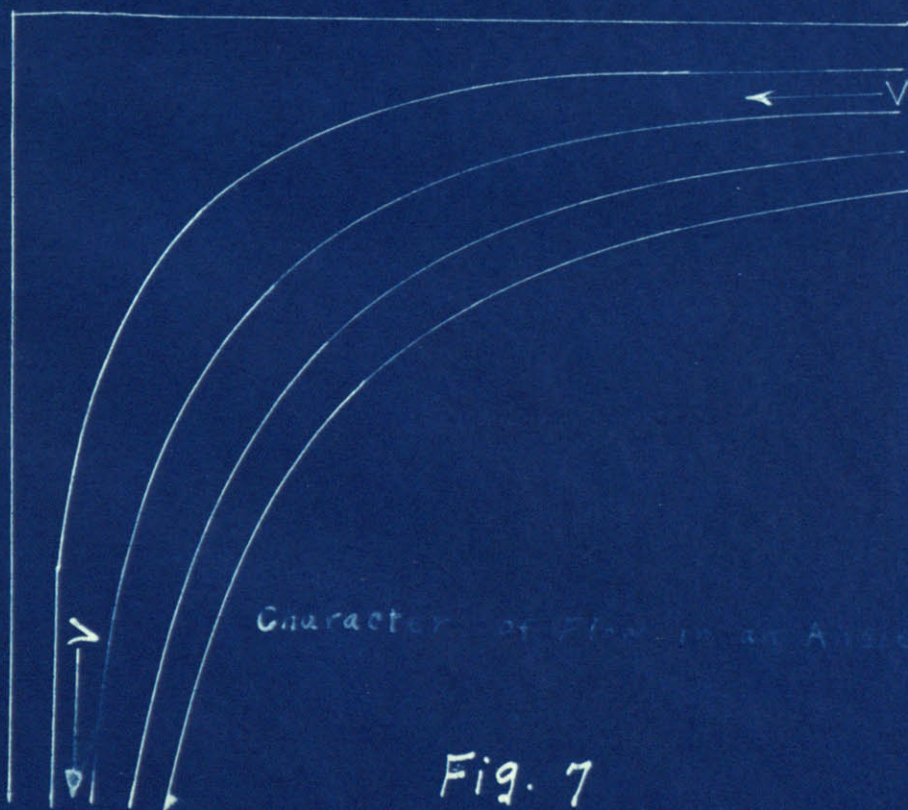


Fig. 7

seems quite unlikely and a little study of the situation soon suggests the need of other methods of solution. The beginner often has difficulty in understanding why the methods of analysis commonly used are necessary.

If the wing shown in figure 6 had a straight line chord from the farthest forward point to the trailing edge shown by the dotted line, it is quite evident that the curvature of the stream underneath would be in the same direction under the wing as forward and the pressure would always be greater than the static pressure of the fluid and consequently the pressure gradient from the leading edge back would be low while on top, the pressure gradient must be steep since it reduces to the static pressure at the point of inflection of the curve a very short distance from the region where the pressure is the same for both upper and lower streams and continues to be reduced in another short distance to considerably below static pressure showing conclusively that the acceleration and velocity must be higher along the top than along the bottom surface.

Referring now to formula 3 page 7, it will be seen that the net lift is proportional to V^2 times a length of arc which might be written $K \int v^2 ds = K \int v^2 r d\theta$, V being changed to v meaning that the direction of the velocity is variable. If r be considered positive when below the horizontal, i.e. when the surface is convex upward and this integral taken around the profile, it is evident from

the preceding paragraph and the diagram, fig. 6 that a net force upward (not necessarily vertical) will result and that the speed need not be considered constant; in fact the higher speed together with the greater distance and smaller radius of curvature along the upper surface would all emphasize the fact that there would be a net lift upward.

II.

PRESSURE ON THE SURFACE BY BERNOULLI'S EQUATION

Pressure on the Surface by Bernoulli's Equation

Equation 3 page 7 for a differential arc can be accurately obtained from Bernoulli's equation. Consider a small mass adjacent to the surface between two stream lines figure 6. The resultant force on the mass is $-dp$ times the cross sectional area of the tube $-Sdp$ and the mass is $D S ds$ and :

$$-Sdp = SDds \frac{d^2 s}{dt^2} = S D ds \frac{dv}{dt} = S D v dv \text{ and } -P = \frac{D}{2} v^2 + C$$

The net pressure is $\int p ds$ around the profile and since the constant would integrate to zero around a closed curve, the net pressure would be $= \frac{D}{2} \int v^2 ds$ which would mean a lift of this amount and for the reasons stated in the preceding paragraph would have a vertical component.

One must not overlook the fact that a non viscous fluid cannot exert a negative pressure and attempt to apply Bernoulli's equation to fluids with speeds greater than $\sqrt{\frac{2C}{D}}$. C can be determined by placing $v = 0$ in the above equation. C = static pressure, i. e. the pressure of the fluid not in motion and standard atmospheric pressure is about 15 lbs per square inch and standard density about .08 lbs per cu. foot so that pressure would

be a minimum or zero when: $V = \sqrt{\frac{30 \times 144}{0.8} g} = 12 \sqrt{375g} = 233 \sqrt{g} = 1350$
feet per sec. or 930 M.P.H.

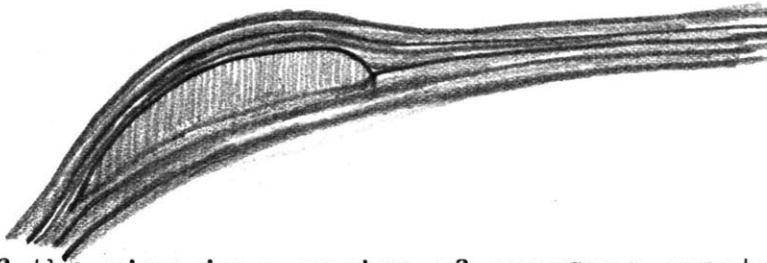
III.

CIRCULATION

Circulation

A quantity called circulation $= \oint v \, ds$ around the profile comes into the mathematics of this subject when developed from the classical hydrodynamics through the Jowkowski transformation which due to the way in which it is introduced is sort of a mysterious quantity with an unexplained origin and without any excuse for an existence except that it is necessary to account for the lift actually obtained in practice. Its origin is usually attributed to viscosity. By the discussion of the next few paragraphs and a look at figure 6, it is obvious that the distance over the top surface is greater, and it looks very plausible that the speed is also greater than along the under surface from the point of zero velocity in front to the trailing edge and it certainly is evident that the integral of $v \, ds$ around the profile would not in general be zero. This fact is not in any manner the result of viscosity but only due to the lack of symmetry of the wing with reference to a line in the direction of the undisturbed flow, in fact the less the viscosity may be, the more pronounced will be the higher velocity above.

To emphasize this important point, so generally misunderstood, it will be well to repeat what has already been said with the aid of the sketch below.



Forward of the wing is a region of pressure greater than the static pressure of the fluid because of the centrifugal force of the fluid forced to turn out of a straight path to get by the wing. This pressure will effect the stream which is destined to pass over the wing exactly as it does the stream which passes beneath except very near the leading edge. At some point on the surface near the forward most point of the wing will be a stagnation point where the fluid is not flowing around either way. That which passes underneath would, at most have only a slight reversal of curvature for a short distance and in most cases would curve in the same direction until past the wing so that the pressure would be greater than the static pressure of the fluid at all points and the pressure gradient from the stagnation point or point of highest pressure at the leading edge toward the trailing edge would be very low, while on the upper surface, the pressure decreases to the value of the static pressure at the inflection point a very short distance from the stagnation point and to considerably below in another very short distance so that the pressure gradient is very steep and the acceleration and velocity much higher over the greater part of the upper surface than beneath in addition to the fact that the distance is also greater.

The above does not determine where the forward stagnation point will be, but it will be determined by the shape of the wing and not be in any way dependent upon viscosity.

IV.

DETERMINATION OF LIFT IN TERMS OF CIRCULATION.

Determination of Lift in Terms of Circulation

In what precedes, we have come twice to the integral of $V^2 dS$ in attempting to determine the lift, but only found from this that there would probably be a lift and that other means for its determination would at least be very desirable. The common method of determining lift, probably originally due to Joukowski is to compute the total change of momentum of the fluid inclosed by a large curve surrounding the wing which is equal to the force upon the wing plus the pressure on the boundary of the region. The result comes out as a vertical force equal to the density times the undisturbed velocity times the circulation. Then the Joukowski transformation is a method of showing how much the circulation would be when the flow follows the profile to the trailing edge as a perfect fluid would until the velocity becomes so great that the centripetal force required to hold the fluid on the top surface exceeds the static pressure plus the cohesive force with which the fluid sticks to the surface, the latter being much reduced by viscous drag in a real fluid.

Ordinarily in the Joukowski transformation, circulation about a cylinder in a perfect fluid of the proper strength to make the flow around the trailing edge of the transformed airfoil zero is assumed and of course its origin about the cylinder is hard to explain, but in

the reverse process, it is easy to see how the airfoil produces a circulation which really means nothing except that it changes the velocity of the stream from a uniform velocity in a straight line to some other velocity which may be considered as a combination of the uniform straight line velocity and some added velocity and this resultant velocity integrated around the curve could only be expected to be zero when absolutely symmetrical. If the actual velocity around the curve were known at all points, the constant velocity could be subtracted and what is left over would be the circulation and for very high angles of attack would actually be a flow toward the trailing edge on top and toward the leading edge below as can easily be seen in water flowing at sufficiently slow speeds so that viscosity is unimportant. The flow underneath is slower and the flow above faster than the main stream. Knowing then that the integral of $v \, dS$ or circulation is not zero except in symmetrical flow and that the lift found by the method to be shown below is proportional to the circulation and having the Joukowski transformation to determine the value of the circulation there is little left to be desired except to know how these determinations are made. It should be

stated, however, that the circulation is a constant independent of the curve around which it is taken if the fluid motion is non rotational and that a perfect fluid

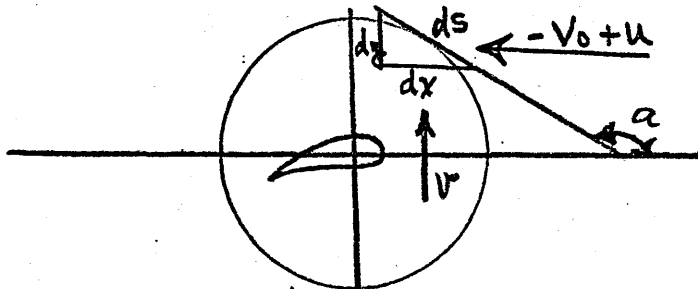
cannot be made to rotate. This is ordinary hydrodynamic theory and will not be discussed in this paper.

Instead of the integral of $v^2 ds$ already found by two methods to give the resultant force on the wing, it is desired to express the force in terms of the integral of $v ds$ which can easily be done by considering the velocity to ^{be} made up of two velocities, the velocity of the undistributed stream - V_0 plus an added velocity u in the direction of X and a velocity V in the direction of y . The velocity is usually considered as the derivative of a potential of velocity, the potential being $-V_0 X + f$, f being a function of the coordinates and the velocity parallel to the x axis being $-V_0 + \frac{\partial f}{\partial x}$ and that parallel to y , $\frac{\partial f}{\partial y}$. As it is not desired to go into a discussion of velocity potential at this point, it will be more logical to use u and v instead of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ although both expressions mean simply the difference between the actual velocity and what the velocity would be if the wing were not present to cause a change.

Now, since at some distance from the wing where the disturbance is small, u and v are of such a magnitude that their product and squares are so small as to be negligible compared to the first powers, the integral of the Bernoulli pressure around a curve will contain only the square of the constant velocity V_0 and the products of V_0 with u and v and the resultant of this pressure and the

and the force of the wing will be equal to the rate of change of the momentum of the fluid within the curve considered.

Consider first the horizontal components and let X be the horizontal force on the wing. The amount of fluid



entering the region in unit time as can be clearly seen from the sketch is

$$\int_0 D(-V_0 + u) ds \sin \alpha + Dv ds \cos \alpha, \quad ds$$

being taken as positive in the direction of the arrows.

The horizontal component of the momentum entering per unit of time is: $\int_0 D(-V_0 + u) [(-V_0 + u) \sin \alpha + v \cos \alpha] ds$.

The horizontal component of the pressure on this boundary is,

by Bernoulli's equation equal to $-\int_0 [H - \frac{1}{2} D \{(-V_0 + u)^2 + v^2\}] \sin \alpha ds$

$$\text{Then } X = \int_0 (H - \frac{1}{2} D V_0^2 + D V_0 u - \frac{1}{2} D u^2 + \frac{1}{2} D v^2) \sin \alpha ds + (D V_0^2 - 2 D V_0 u + D u^2) \sin \alpha ds + (-D V_0 u + v u) \cos \alpha ds$$

and since the constant terms times a sine or cosine and

ds integrate to zero around the curve, they may be discarded

with the second order terms and so $X = -D V_0 u \sin \alpha ds + v \cos \alpha ds$

which is the integral of the fluid entering the region due to

the velocities u and v but since the density is considered

as constant, the amount of fluid in the region is constant

and this integral is zero, meaning that in a perfect or

non viscous fluid, there would be no force in the direction

of motion.

The lift is computed in the same manner:

$$Y = \int_0 (H - \frac{1}{2} D V_0^2 + D V_0 u - \frac{1}{2} D u^2 + \frac{1}{2} D v^2) \cos \alpha ds + D(-V_0 v + u v) \sin \alpha ds + D v^2 \sin \alpha ds$$

$$= D V_0 \int_0 (u \cos \alpha - v \sin \alpha) ds$$

and $u \cos \alpha - v \sin \alpha$ is
the projection of the variable part of the velocity on
the circumference of the curve and the integral is the
circulation, so the lift is equal to the density times the
velocity of the undisturbed stream times the circulation.

Determination of the Amount of Circulation

We are greatly indebted to Professor Joukowski of the University of Moscow for a method^{of} determining the amount of the circulation about a wing and consequently the lift as shown above.

Professor Joukowski assumed the circulation and developed the wing profile accordingly. The Joukowski formula transforms a circle into an airfoil and a suitable amount of circulation around the circle was assumed so that there should be a stagnation point on that part of the circle which is transformed into the trailing edge and then when the circle with its surrounding flow pattern is transformed into the airfoil with the surrounding flow pattern, the stagnation point occurs at the proper place. On account of the development having been made in this manner, one is inclined to attempt to account for the circulation about the circle, but to understand the situation, one should consider the circulation about the wing as transformed into the circulation about the circle for more convenient study. The stagnation point occurs at the trailing edge of the wing simply because, in a perfect fluid there is no force that could be applied to make the fluid pass around the trailing edge. If the trailing edge were sharp, this would require an infinite centripetal force. The flow over

over the top will follow the contour until the radial acceleration necessary is greater than the pressure plus the force with which the fluid sticks to the surface can supply and then it leaves a partial vacuum occupied only by turbulent fluid dragged along with the wing and in air by a certain fraction which due to its molecular speed in that direction has sufficient speed along the radius of the curve. Due to this reduced pressure, the stream underneath is bent partly round the trailing edge and in a viscous fluid considerable is dragged clear around. For all speeds below this critical speed, the stagnation point is on the trailing edge for lack of any force to fix it elsewhere and it needs no other explanation.

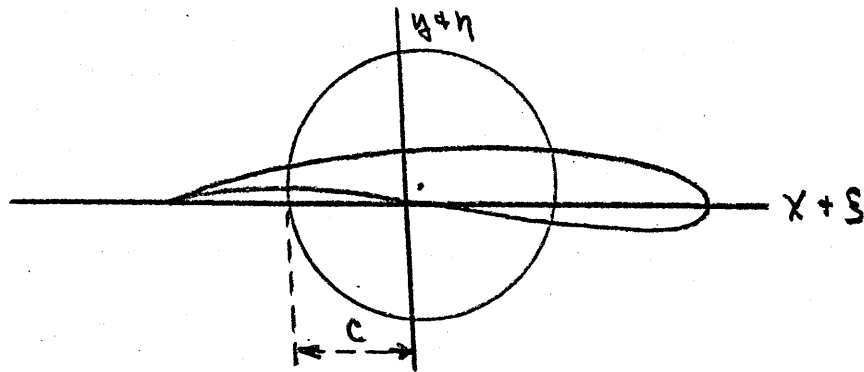
If a vector Z locates any point in the $X Y$ plane and the X component be designated by X and the Y component by $\sqrt{-1} y = i y$, z is the vector sum of its components $= X + i y$, an ordinary complex number, the properties of which will not be discussed except where necessary. The Joukowski formula locates corresponding points by another vector $\zeta = z + \frac{c^2}{z}$ and by this formula, a circle with its center slightly to the right and above the origin of the $x y$ axes can be transformed into a suitable wing section by simply taking the vector z to any point of the circumference of the circle, calculating ζ and locating the corresponding point on the wing section. Stream lines past the circle can also be transformed to stream lines past the wing and it is

important to note, that at large distances from the wing, ~~the "S"~~, for large values of z , the stream lines will not be changed which is as it should be. Having a wing designed by this method, its corresponding circle can be found and of course the point on the surface corresponding to the trailing edge where the stagnation point should be, and then it is only necessary to determine an amount of circulation which when combined with the steady flow of the undisturbed stream (which is the same for both the circle and the wing), will fix the stagnation point at the proper place. The wing having been developed by this process, however, means that these data are obtained in the reverse order.

In order to show how the circulation and the steady flow is combined, it will be necessary to digress for a brief review of the classical hydrodynamic method of determining the flow pattern about the circular profile and while digressing, a few of the important characteristics of the Joukowski transformation will be mentioned.

The function, $\zeta = z + c^2/z = x + iy + c^2/(x + iy)$
 $= x + iy + c^2 (x - iy) / (x^2 + y^2) = X \left[1 + c^2/(x^2 + y^2) \right]$
 $+ i y \left[1 - c^2/(x^2 + y^2) \right]$ commonly written as $\zeta + i\eta$.

Referring to the sketch below, C is the value of X where the circle intersects the x axis and when X has this value and $y = 0$ the equation shows that $\zeta = 2x = 2c$ and $\eta = 0$



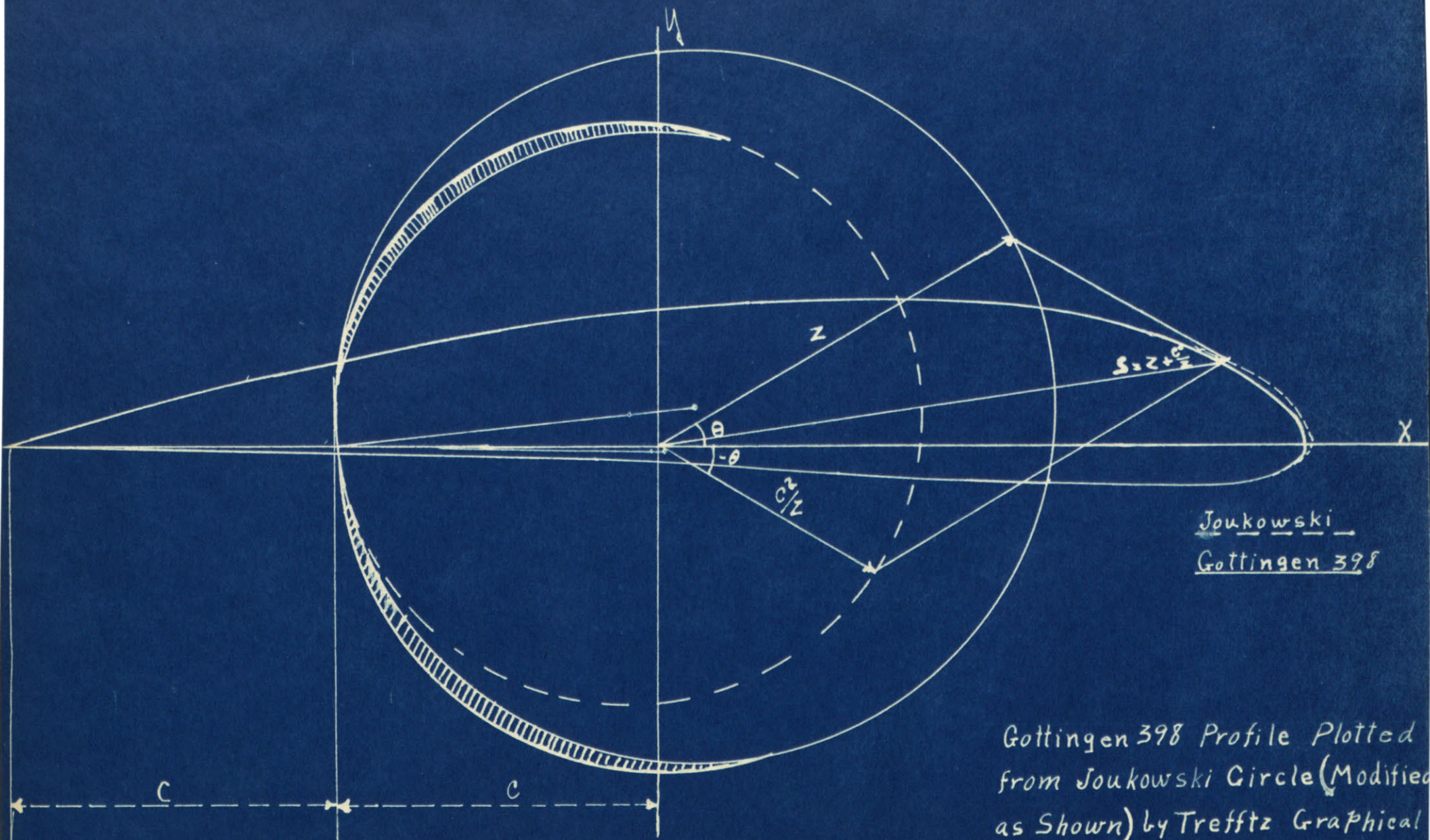
and for the right hand intercept, x is just slightly greater than c so ξ is slightly less than $2x$ and n has a very small positive value. The equation also shows that the points of intersection of the circle and the y axis where $x = 0$ stay on the axis since ξ is also zero. From the sketch, which would make about the average wing profile, it can be seen that n would be equal to y multiplied by $(1 - c^2/y^2)$ which would be very much shorter than the y ordinate since y is only a little greater than c for the upper intercept and might be greater or less or equal but very nearly equal to c for the lower intercept. Other points are less easily transformed but by computing a very few, it can be seen that the ξ curve would take the form of a wing section. A little more computation is necessary to show it, but the upper and lower curves of these profiles are always tangent at the trailing edge. Modifications of the Joukowski formula have been made to give the trailing edge an angle to facilitate actual construction.*

A very simple graphical method for drawing Joukowski profiles which lends itself well to mechanical tracing devices is described by E. Trefftz in Z. F. M.

* Suggested by Karman & Trefftz, Z.F.M. 1918. See also R.M.911 Aeronautical Research Committee of Great Britain by Glauert and Glauert's book on "Airfoil and Airscrew Theory" p. 78.

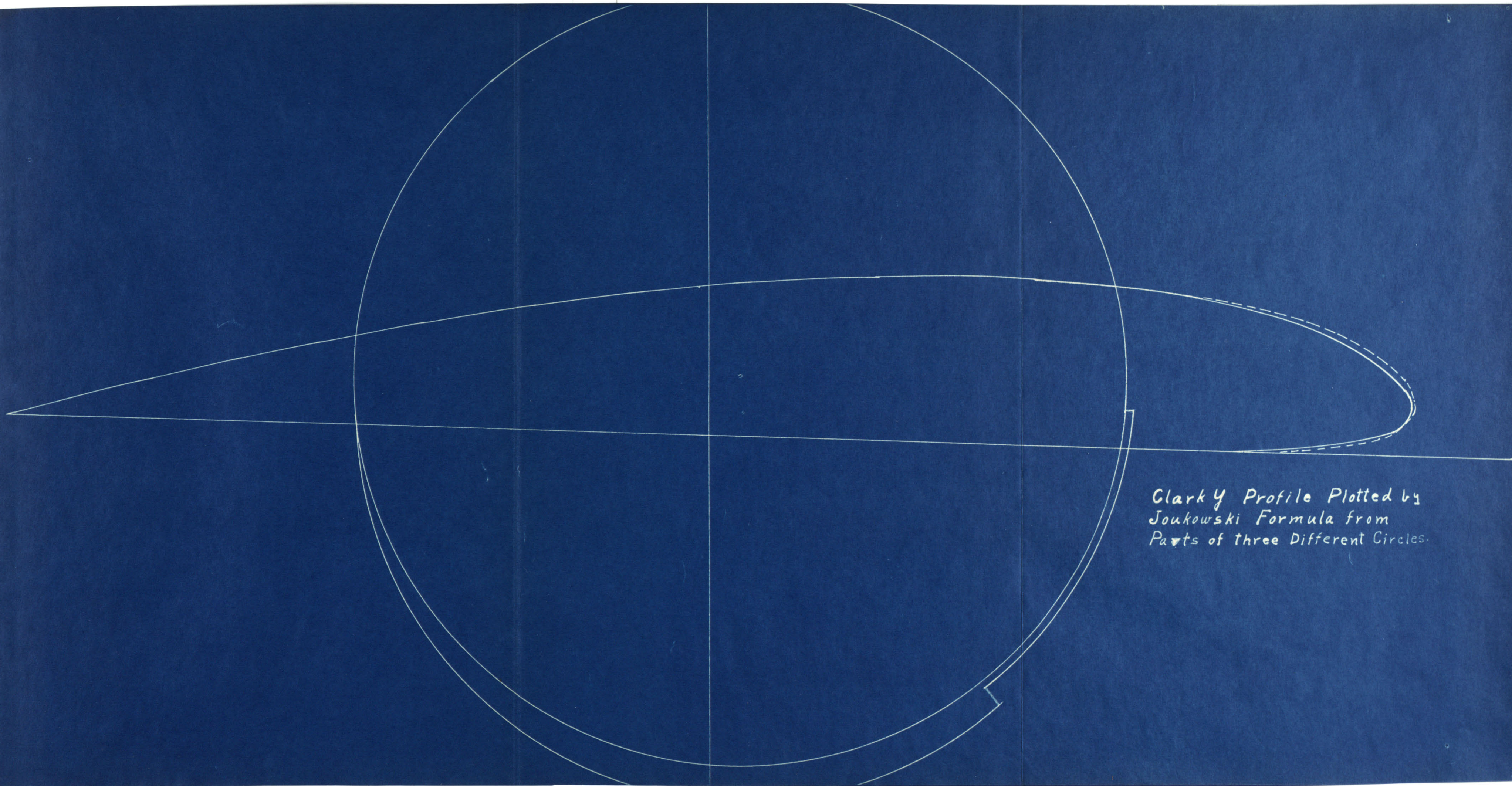
May 31, 1913 and translated by the N. A. C. A. in "Technical Memorandum" #336. This method can be best understood by noting a few further characteristics of the complex variable, z . The vector $z = x + iy$ can as well be described by its length r and the angle θ it makes with the x axis; $x = r \cos \theta$ and $y = r \sin \theta$ so $z = r (\cos \theta + i \sin \theta) = re^{i\theta}$ and $c^2/z = (c^2/r)e^{-i\theta}$ and $\bar{z} = re^{-i\theta} (c^2/r)e^{i\theta}$ and can be easily found by the parallelogram law as the sum of two vectors of lengths r with a positive angle θ and c^2/r with a negative angle θ . When $r = c$ and $\theta = \pi$ the two vectors coincide so the circle described by z and the circle described by c^2/z both pass through the same point and they are tangent at this point so the center of the c^2/z circle lies on the radius of the z circle.

If X_1 be the point at which the z circle crosses the positive x axis and X_2 the point of intersection of the x axis with the c^2/z circle, y being zero, $X_2 = c^2/(x_1 + i y_1) = c^2/x_1$ so with the two points of crossing the x axis and the line on which the center lies, the c^2/z circle is determined and the pair of vectors determining the profile can be drawn for as many values of θ as may be necessary to determine a smooth curve. The Trefftz method is slightly simplified because he completes a triangle on the two vectors and bisects the third side for the points on the wing curve instead of completing a parallelogram.



Joukowski
Gottingen 398

Gottingen 398 Profile Plotted
from Joukowski Circle (Modified
as Shown) by Trefftz Graphical
Method



Clark Y Profile Plotted by
Joukowski Formula from
Parts of three Different Circles.

V.

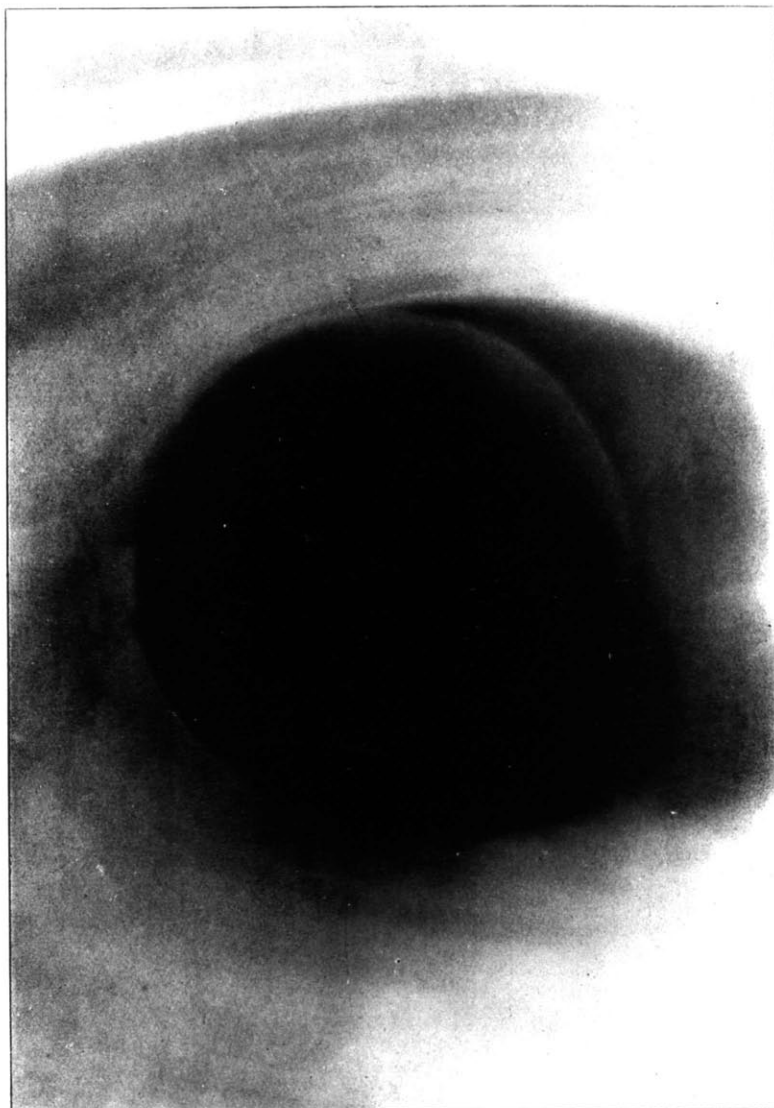
FLOW PATTERN ABOUT A CYLINDER

Flow Pattern about a Cylinder.

As mentioned in V, in order to fix the stagnation point of the flow at the point on the circle which by the Joukowski transformation becomes the trailing edge of the wing, the flow pattern about the circle must be known and then the necessary amount of circulation can be determined and this circulation will be the amount actually produced by the wing when the flow is everywhere tangential to the surface as for speeds not too high.

To get this flow pattern, a very highly artificial device is used which, however, represents the facts very well for slower speeds and forward of the wing for higher speeds but fails except in front, at speeds where viscosity becomes important. For a cylinder, this device shows pressures in rear exactly equal for points symmetrically located to the pressures at various points in front while, in fact there may even be a vacuum behind and a high pressure in front. (See attached photo copied from the Frontispiece of Lanchester's "Aerodynamics.")

If one imagines a "source" in a large volume of fluid where fluid is created as might be well approximated in two dimensions by passing a circular pipe perpendicularly through a body of fluid confined between two parallel planes and allowing fluid to discharge radially from the pipe between the two planes, it is reasonably evident that the

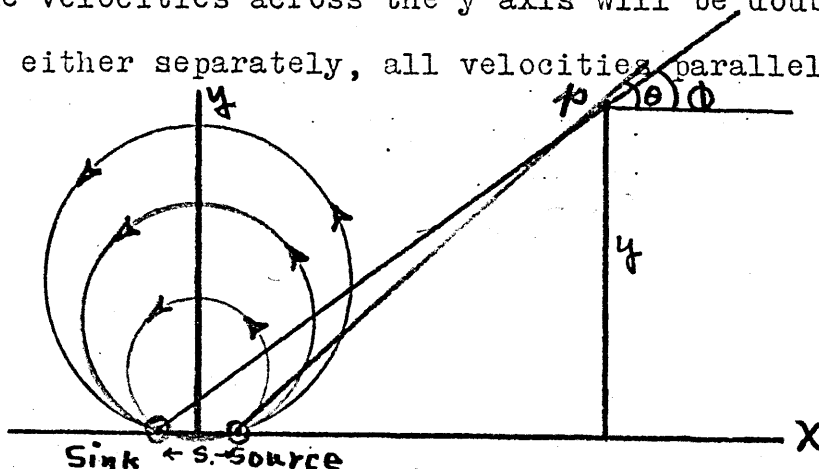


PHOTOGRAPH SHOWING FLOW OF AIR ROUND A CYLINDER IN MOTION.

From Lanchester's "Aerodynamics"

fluid will pass out into the main body of the fluid with a velocity everywhere in the direction of the radii of the pipe section and with a speed inversely proportional to the distance from the center of the pipe. If the volume of fluid produced per unit of time is m per unit of thickness of the body of fluid considered, then an amount m must pass across the circumference of each circle concentric with the pipe in unit time so that the radial speed would be $m/2\pi r$, r being the radial distance from the center of the pipe.

If instead of a source, we consider a "sink" or a negative source, where fluid is discharged from the main body of fluid into the pipe, the situation would be just the reverse, with velocities of the same magnitude but in opposite directions. If now, a source and a sink are both present and near together, the velocities everywhere will be the vector sum of the velocities due to each separately. If the x axis is taken through the source and the sink and the y axis through a point midway between, the vertical velocity along the y axis will be the sum of two velocities of equal magnitude but oppositely directed and will be zero, while the velocities across the y axis will be double that due to either separately, all velocities parallel to the x axis



directed toward the y axis are increased and those directed away from the y axis are decreased and all velocities parallel to y are decreased so that the final result is a flow in closed curves as shown in the sketch, and proved in equation 6 below.

If the distance between the source and sink is s, the velocity at any point p at (x, y) has a component outward along the line from the source equal to

$$v_1 = m / (2\pi \sqrt{y^2 + (x - \frac{s}{2})^2}) \text{ and if the distance } s \text{ is so small that } s^2/4 \text{ can be neglected, } v_1 = m / (2\pi \sqrt{y^2 + x^2 - sx}).$$

The other component v_2 of the velocity at p is inward along the line to the sink and equal to $-m / (2\pi \sqrt{y^2 + x^2 - sx})$.

The resultant velocity v has an x component $= v_1 \cos \theta + v_2 \cos \phi$

and a y component $= v_1 \sin \theta + v_2 \sin \phi$ and $v^2 =$ the

sum of the squares of the rectangular components

$$= v_1^2 \sin^2 \theta + v_2^2 \sin^2 \phi + 2v_1 v_2 (\sin \theta \sin \phi + \cos \theta \cos \phi) + v_1^2 \cos^2 \theta + v_2^2 \cos^2 \phi$$

$$= v_1^2 + v_2^2 + 2v_1 v_2 \cos(\theta - \phi) \text{ but since } s \text{ is very small, } \cos(\theta - \phi)$$

can be taken as 1 and $v^2 = v_1^2 + v_2^2 + 2v_1 v_2$, on substitution,

$$v^2 = \frac{m^2}{4\pi^2} \left(\frac{1}{y^2 + x^2 - sx + s^2/4} + \frac{1}{y^2 + x^2 + sx + s^2/4} + \frac{2}{\sqrt{(y^2 + x^2 - sx)(y^2 + x^2 + sx)}} \right)$$

$$= \frac{m^2}{4\pi^2} \left(\frac{y^2 + x^2 - sx + y^2 + x^2 + sx + s^2}{y^2 + x^2 + 2x^2 y^2 + y^2 sx - y^2 sx + sx^2 - sx^2} \right) = \frac{m^2}{4\pi^2} \left(\frac{2s^2}{(y^2 + x^2)^2} \right)$$

$$= \frac{2m^2 s^2}{4\pi^2 r^4} \text{ and } v = \frac{ms}{2\pi r^2} = \frac{K}{r^2}$$

Where r is the distance from the origin of the axes to the

point. The tangent of the angle that this combined flow

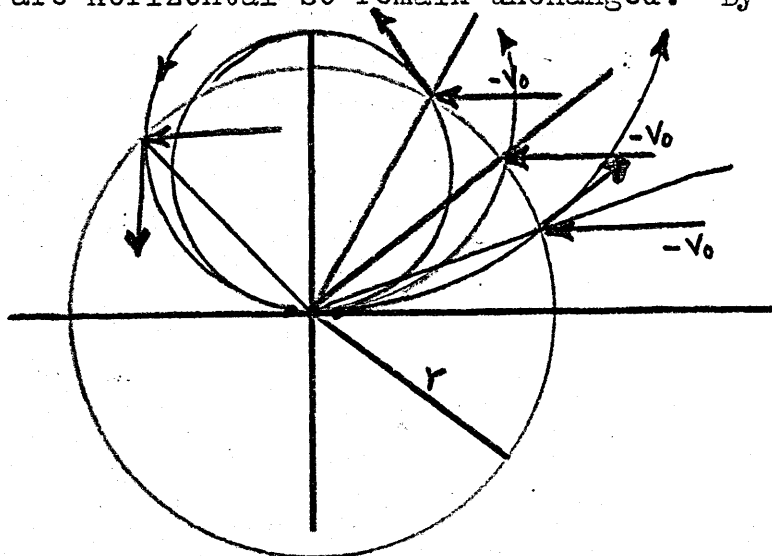
$$\text{makes with the x axis is } \frac{v_1 \sin \theta + v_2 \sin \phi}{v_1 \cos \theta + v_2 \cos \phi}$$

$$= \frac{\frac{y}{y^2 + x^2 - sx} - \frac{y}{y^2 + x^2 + sx}}{\frac{x - s/2}{y^2 + x^2 - sx} - \frac{x + s/2}{y^2 + x^2 + sx}} = \frac{y(y^2 + x^2 + sx - y^2 - x^2 - sx)}{(x - s/2)(y^2 + x^2 + sx) - (x + s/2)(y^2 + x^2 - sx)}$$

$$= \frac{2sxy}{2sx^2 - sx^2 - sy^2} = \frac{2xy}{x^2 - y^2} \quad (6)$$

The flow is parallel to the x axis where $y = 0$ and vertical where $x = y$ and parallel to x again where it crosses the y axis.

If this doublet (source and sink) be placed in a steady stream of velocity $-V_0$ parallel to the x axis, there will be one value of r for which the velocity of the doublet is V_0 and this will just neutralize the superimposed stream at a point on each side of the origin symmetrically located on the X axis. At angles of 45 degrees from the origin where the velocity of the doublet is vertical as shown in equation 6, the resultant velocity is at 45° or perpendicular to r and at 90° both velocities are horizontal so remain unchanged. By simple geometry,



it can be shown that around a circle with center at the origin, the velocity of the doublet and that of the

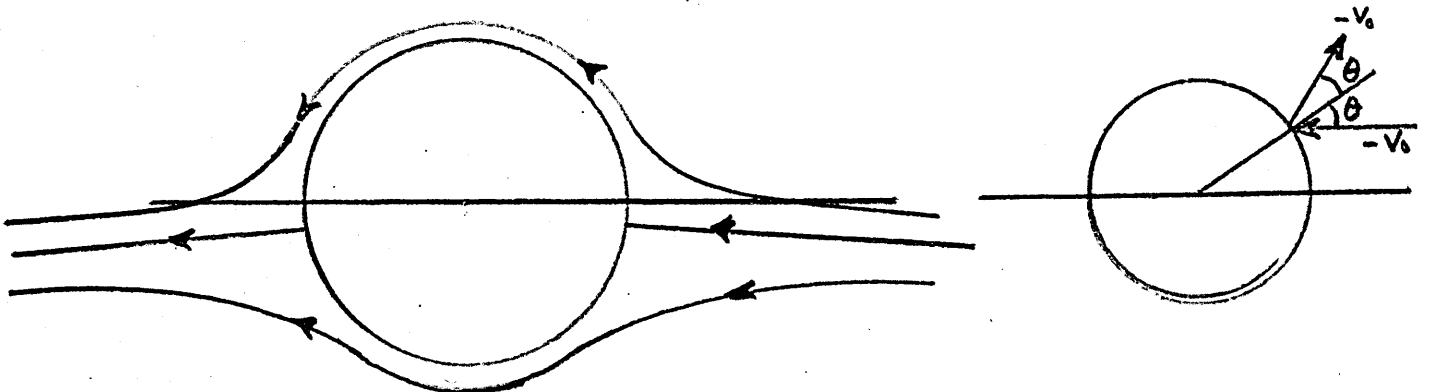
uniform stream make equal angles with the radius and so for the particular radius where the velocity of the doublet is equal to V_0 , the radial components of the two velocities just neutralize each other and the resultant flow is along

the circumference of the circle and no fluid crosses this circle. Since no fluid crosses this circular boundary, it can be replaced by a solid boundary or cylinder without changing the outside flow and if this substitution were made, it can be seen from the diagram above that the flow is reflected off the cylinder at every point with an angle of incidence equal to the angle of attack as would be expected on the outside of the right hand semicircle and the inside of the left hand semicircle quite analogous to the reflection of light, but in the case of light, we would certainly expect a shadow to the left of the cylinder, and with actual fluids we obtain a similar effect which might be called a "fluid shadow." On the right a force supplied by the surface of the cylinder corresponds to the force of the fluid from the source, but on the left it is not so easy to visualize the inflow toward the sink after the sink has been removed and it must be explained in terms of the pressure gradient of the fluid resulting from the velocity gradient. The difficulties of this explanation are greatly magnified by experimental contradiction.

A question as to the value of m might arise since the radius of the cylinder was determined by m and V_0 but having a given cylinder, the value of m to use for any computations of the velocity would be that corresponding to the radius of the cylinder since $m = 2 \pi r V_0$.

no transformation in
effect eliminates the
velocity effect.

In spite of the difficulties mentioned in the second preceding paragraph, mathematicians substitute the cylinder for the doublet and continue the development with a flow symmetrical with respect to both axes, and then for the Joukowski process in which a flow non symmetrical with respect to the X axis is necessary, a circulating flow around the cylinder is assumed, i. e. a flow in circles concentric with the cylinder with a tangential velocity inversely proportional to the radius so that the integral of the velocity around the circle is a constant regardless of the distance around. Then $2 \pi r v = k =$ the circulation as defined in III and if this new velocity is added to that already described as due to the doublet and the uniform flow of the cylinder and the uniform flow, an unsymmetrical flow will result. Assume the circulation to be going counter clock wise and it is evident that the velocity above will be increased and that below decreased and that the stagnation points will be below the x axis, roughly as shown in the sketch below. This is the result needed to carry out this



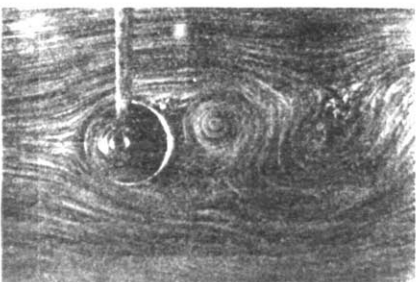
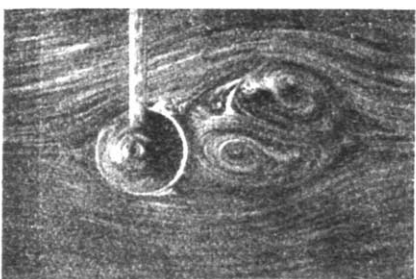
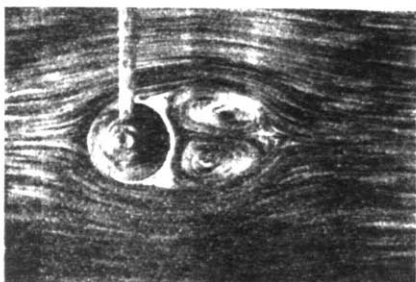
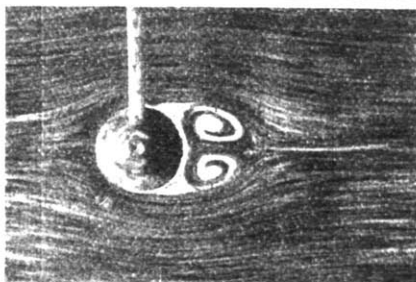
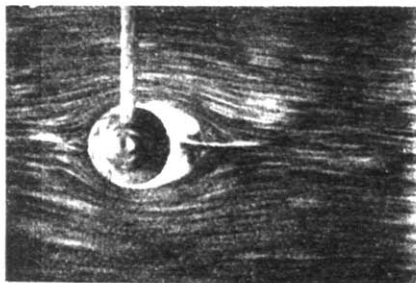
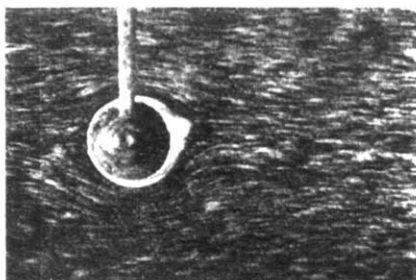


Abb. 6. $u \cdot r = 0$.

Strömungszustand an einem
Zylinder zu Beginn der
Strömung.

Abb. 6:
Nicht rotierender Zylinder.

Abb. 7:
Rotierender Zylinder, wobei
die Umfangsgeschwindigkeit
gleich 4facher Strömungs-
geschwindigkeit ist.

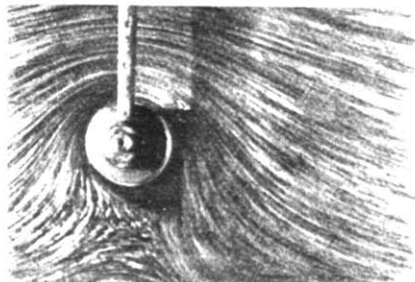
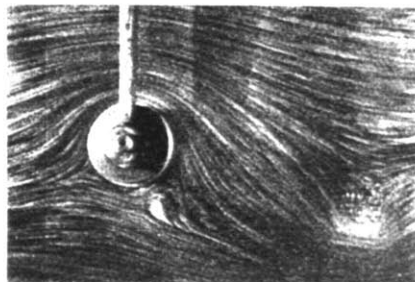
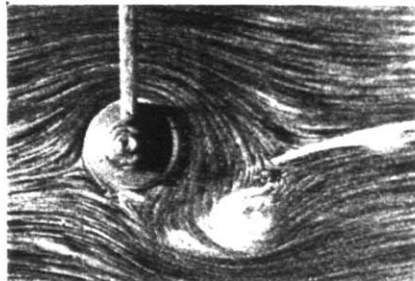
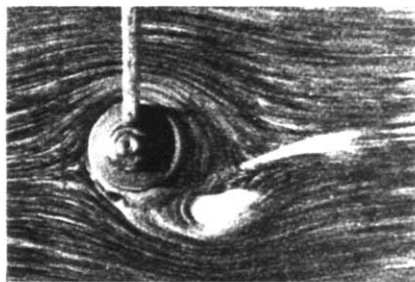
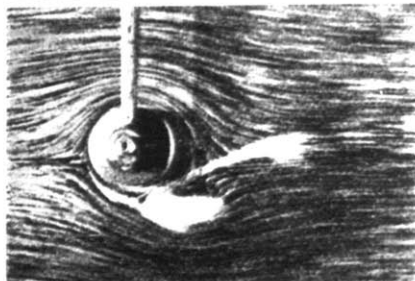
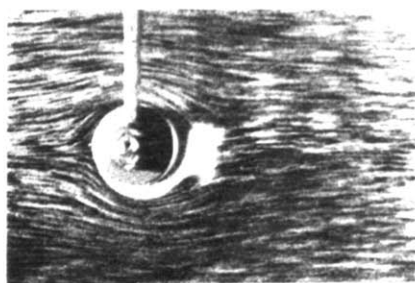


Abb. 7. $u \cdot r = 4$.

Joukowski transformation with a stagnation point on the trailing edge of the wing. The steady stream at each point has a component $= V_o \sin \theta$ along the circumference and since the outflow from the sink makes an angle θ with the radius, it also has a component $= V_o \sin \theta$ so that the total circumferential velocity due to the circulation is $K/2\pi r$ the stagnation point in front will be at an angle below the x axis such that $K/2\pi r = 2V_o \sin \theta$, $\theta = \sin^{-1} K/(4\pi r V_o)$ and the rear stagnation point will be symmetrically located.

Summary of Steps Necessary to Determine Lift.

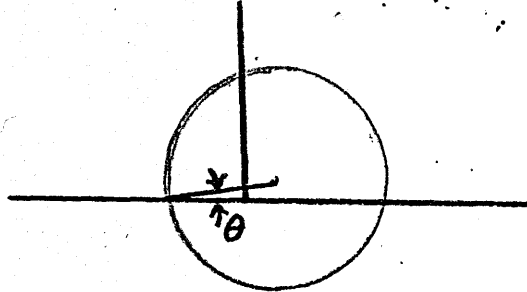
The momentum created per unit time in a large region surrounding the wing was found by integrating this momentum crossing the boundary per unit time and this equaled to the force of the wing plus the pressure as determined by Bernoulli's equation integrated around the curve. Thus the force on the wing was found to be vertical and equal to $\rho V_0 \int_0^s v ds$ where v is the velocity along the boundary and ds an element of the boundary curve.

Since the determination of the integral of $v ds$ cannot be made directly, it is found by the Joukowski transformation by which the wing if originally designed by this process can be transformed into a cylinder with the trailing edge definitely located on the cylinder. Around this cylinder the flow due to the uniform stream is determined by means of an artifice, the source and sink*, and to the velocity thus found on the circumference of the cylinder is added a circulating velocity sufficient to place one stagnation point on the trailing edge.

The best method of determining the Joukowski profile is by the Trefftz graphical scheme and the trailing edge is the point of intersection of the circle and the x axis and the angle θ used in determining the circulation

*This result can be obtained without this artifice from the function : $F = -V_0 Z - \frac{k}{Z}$.

is the angle between the x axis and a line from the center of the circle to the trailing edge. $K = 4\pi r V_0 \sin \theta$



VI.

A POSSIBLE METHOD OF CALCULATING LIFT DIRECTLY

BY BERNOULLI'S EQUATION.

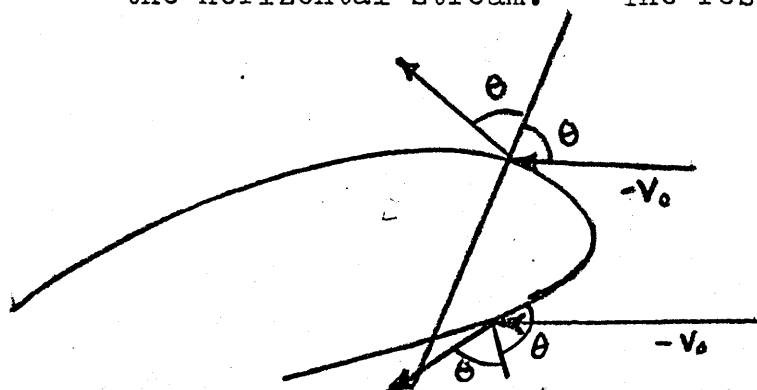
A Possible Method of Calculating Lift Directly
by Bernoulli's Equation.

From the analysis of the flow of a source and sink and the equivalent cylinder in a uniform stream (VI page 26), the resulting flow ahead of the cylinder was just as though the steady stream were first considered as undisturbed until it comes in contact with the surface and then were reflected off with an angle of incidence equal to the angle of attack at every point of the surface just as would be the case with light or elastic particles and then the actual velocity at any point ~~is~~ the resultant of the velocity of all the reflected streams which correspond exactly to the streams from the doublet, and the steady stream. This suggests the advisability of analyzing the flow about a wing considering similar reflection from all points of the forward part of the surface.

The reflected stream would have a component along the normal to the surface at each point just neutralizing the radial component of the horizontal stream and each would have a component tangentially to the curve equal to $V_0 \sin \theta$ where θ is the angle between the normal to the surface and the horizontal stream. The resultant velocity along the

curve then is

$2 V_0 \sin \theta$, and where



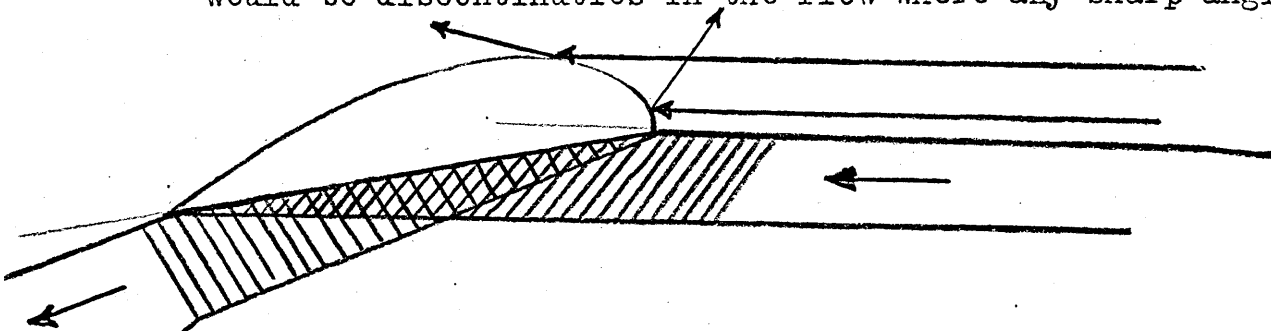
the flow becomes horizontal on top of the wing, the velocity would be $2 V_o$. The front stagnation point would be where the surface is perpendicular to V_o . The velocity below the wing would be $2 V_o$ if the chord were parallel to the direction of motion, but for an angle of attack would be reduced to a value always below $2V_o$.

By Bernoulli's equation, the pressure would be $H - \frac{1}{2}DV^2 = H - 2DV_o^2 \sin^2 \theta$ and the vertical or lift component would be $(H - 2DV_o^2 \sin^2 \theta) \sin \theta$ and the lift on an element of the arc $ds = R d\theta$ would be $(H - 2DV_o^2 \sin^2 \theta) R \sin \theta d\theta$. The drag component would be $(H - 2DV_o^2 \sin^2 \theta) R \cos \theta d\theta$. For a straight line chord the total pressure on the straight section could be computed without integration, and for the curved parts, it would not be difficult to add up the pressure for short elements of the arc using the average value of R and θ for the section. If, however the value of R can be conveniently expressed in terms of θ it may be possible to integrate the pressure, but in most cases this would be more trouble than computing the sections or perhaps using Simpson's rule.

To the rear of the highest point on the top surface, the reflection phenomena would not apply and other means of determining the velocity would be necessary. For flat wings at low angles of attack, it would not vary much from $2V_o$, but the horizontal component would be decreased

and a vertical component added in passing toward the trailing edge.

To make a successful computation by this method would require some definite knowledge of the behavior of the stream after passing the crest of the wing. Computations were made assuming the velocity on this portion, as on the forward section, to be equal to $2V_o \sin \theta$, but the results are much less than experimental results. The assumption that the horizontal component of the velocity remains constant during the time required to pass to the rear of the wing so that when the stream follows the surface, the velocity would be $2V_o/\sin \theta$ gives results too large, but since these results are not far from experimental results and experimental values are between the two, the idea would seem to merit further study. It is clear from the sketch below that there would be discontinuities in the flow where any sharp angles



are involved. It can readily be seen that there would be a sharp line dividing a region into which fluid is reflected and a region in which none is reflected and if there were a sharp curve on the nose, a very narrow stream would have to diverge over the area shown on the sketch without any reflected flow and so the situation would require a further analysis.

VII.

RESISTANCE TO MOTION.

Resistance to Motion

Computations of resistance in a viscous fluid cannot be made by any definite complete method and only partially complete and partially satisfactory estimates are possible even in two dimensions.

To discuss this problem completely is much beyond the scope of this paper and indeed will supply subject matter for investigation for a long time to come. However, a few simpler considerations will be mentioned and the most commonly accepted theory will be very briefly outlined.

As in the foregoing discussion of lift in a non viscous fluid, it will be found very helpful in discussing resistance to first analyze a less complex flow to show why there should be resistance and to get a rough idea of the amount to be expected before discussing the very difficult problem of analysis of the actual conditions. The foregoing discussion of lift, though necessarily very brief is complete for two dimensional flow of a non viscous incompressible fluid from which actual lift could be easily computed. The problem of resistance, however, is a very difficult one and although it has received a great deal of consideration by the most competent investigators, its

solution has never been satisfactorially worked out.

Prandtl (Verhandl. d. III intern. math. Kongress (Heidelberg, 1904)*)

Blasius (Grenzschichten in Flüssigkeiten mit kleiner Reibung - "Zeitschrift f. math. u. phys.". 1908) and Karman (über laminare und turbulente Reibung, Z.A.M.M. 1921) have probably contributed most technically to the subject, though good general discussions of the subject can be found in standard treatises on aerodynamics and hydrodynamics (Lanchester, Cowley and Levy, Lamb etc.)

If a small stream of water with considerable speed is projected against a curved surface as shown in attached photograph, the way in which the entire stream clings to the surface as soon as the lower edge of the stream is allowed to touch the surface is very striking and the force it exerts upon the surface is surprisingly great. A stream $1/16$ of an inch in diameter with a speed estimated at 50 feet per second when projected tangentially to the surface so that the lower part of the stream just makes contact with the surface is entirely stopped in crossing the 10 inch wing shown in the attached photograph. This stream, of course spreads out over an area several times as wide as the diameter of the stream so that if it were projected in a sheet instead of a round stream the amount that could thus be completely stopped would probably not be greater than .005 of an inch in thickness in travel-

*See n.a.c.a. translations for other prandtl papers

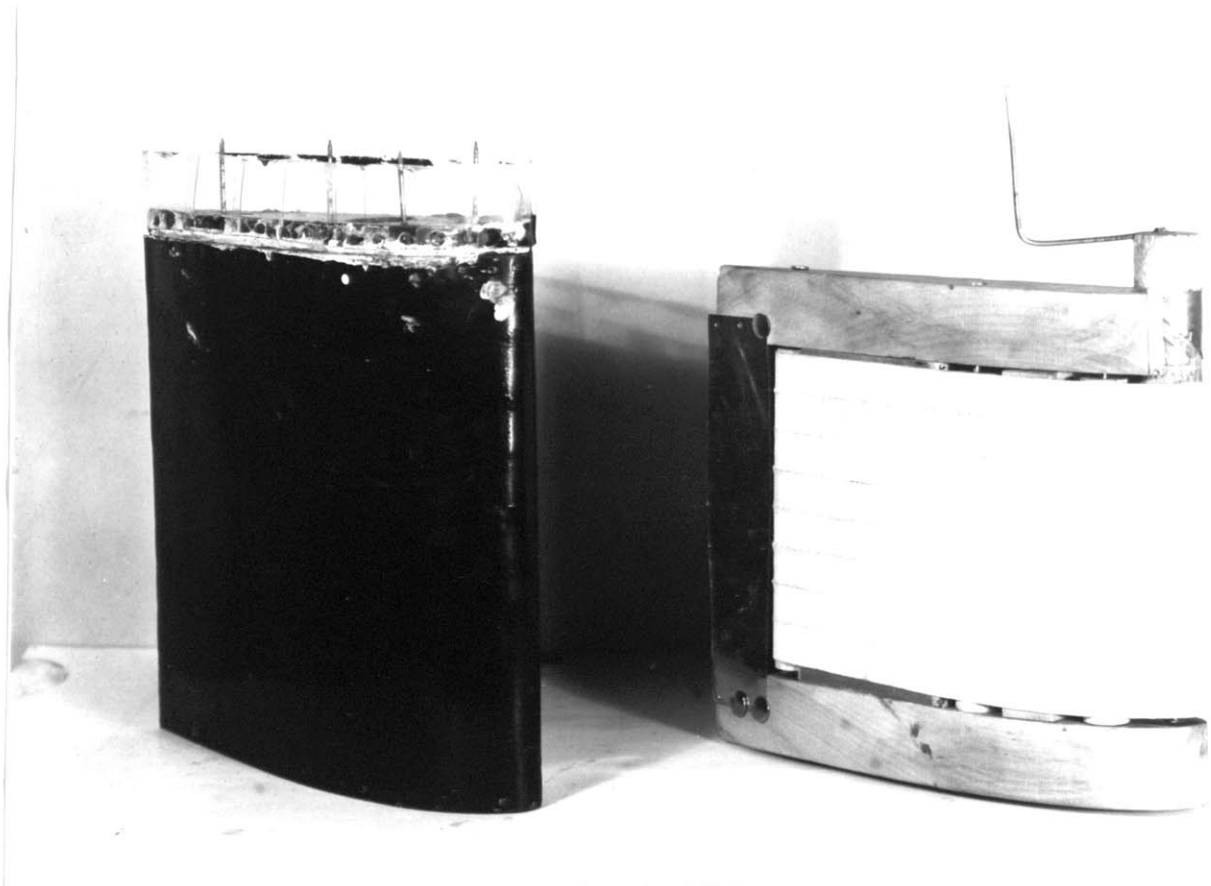


Illustration of Loss of Momentum Due to Sticking to Surface and Methods Suggested to prevent this Loss.

ing across a foot of surface. This would mean that .025 times 50 pounds of water would be stopped each second by every square foot of area and the momentum lost would be 62.5 lb. ft./sec representing a force of about 2 pounds on each surface or 4 pounds in all. This represents a resistance coefficient per mile per hour of .00344 pounds. For a layer of air of the same thickness, the force would only be about .0000043 pounds but the air would probably stick as tenaciously as the water and a layer of much greater thickness would be stopped. Because of the much greater average distance from the surface, it could not be expected that as great a mass of air as of water could be stopped, but it is not unlikely that the full 1/16 inch layer would be stopped which would give a force of .000043 pounds which is in the experimental range.

It is probable that in a continuous stream, the fluid immediately in contact with the surface would lose momentum very rapidly and that the rate of loss would decrease continuously though rapidly at a distance from the surface in accordance with Prandtl's "boundary layer" theory. Prandtl's method resulted from careful study of the equation of motion of a viscous fluid:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \nabla^2 u$$

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial y} + \nu \nabla^2 v$$

Where u , v and w are velocity components parallel to the x , y and z axes and ν the kinematic viscosity coefficient.
(See Glauert "Airfoil & Airscrew Theory" p.111)

Near the surface where the velocity gradient is high the viscosity is important but at some distance away can be neglected and very near the surface, the viscous forces are so great as to justify neglecting other terms and accordingly, the equations of motion can be simplified if judiciously applied and results obtained that could not otherwise be obtained.

The problem of resistance can be attacked by a study of the rotational energy given to the stream so common to daily observation and Karman has investigated the formation of vortices, their stability etc. and shown the existence of the so called "Karman vortex Streets" or curves of discontinuity in certain cases. These curves of discontinuity have been suggested for a non viscous fluid but not readily accepted because of their lack of harmony with mathematical theory. To compute the energy of a single vortex would of course be impossible but the problem can be handled more generally. This resistance is commonly known as the "Form or profile Drag". For a flat plate the form drag is very high because of the strong vortices shed but where the body is such that the stream follows the surface, the drag is small and usually described as "Skin friction". It is more important to know at what speeds and aspects, the form drag becomes important than to know the amount, but Karman has developed formulae from which the approximate coefficient can actually be computed. (See Glauert p. 99 or Karman, loc, cit.).

The wing induces a vertical velocity in the incoming stream so that the lift force which is perpendicular to the stream is not perpendicular to the direction of flight but has a component opposing the motion which being due to the induced velocity is called the "induced drag" and the coefficient of this drag is $\frac{S}{4\pi s} \frac{\sum n A_n^2 k_1^2}{A_1^2}$ for a monoplane aerofoil where S is the surface s the span, k_1 the lift coefficient and the A_n 's are the coefficients of the circulation represented by a Fourier series: $K = 4sV \sum A_n \sin n\theta$

(See Glauert p. 141). The resistance is complicated by so many factors that a less complicated general formula could not be expected but for individual problems reasonably satisfactory data can be obtained.

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